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Theoretical Fundamentals of the Method for Thermal Diffusivity Measurements from Auto-Oscillation Parameters in a System with a Thermal Feedback

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Theoretical fundamentals of the method to determine thermal diffusivity from auto-oscillation parameters in a control system, CS, with thermal feedback through the test specimen, are developed. The equation of a CS with a flat specimen and proportional controller (nonlinear boundary value problem of nonstationary heat conduction) is considered. Periodic solutions of the boundary value problem, which is linearized in the vicinity of the stationary solution, are analyzed. It is proved that, with a certain value of CS gain factor, excitation of auto-oscillations occurs. Their frequency ω_c is related to the thermal diffusivity α as $\alpha = C\omega_c$, where C is constant. By nonlinear analysis, it is revealed that the auto-oscillation excitation mode is soft and the frequency depends on the gain factor to a very weak degree. Formulas for calculation of the thermal diffusivity and the specimen temperature field are obtained.

KEY WORDS: auto-oscillations; nonlinear analysis; temperature wave; thermal diffusivity; thermal feedback.

1. INTRODUCTION

The pulse method and the method of temperature waves [1, 2] are the most widespread ways to measure thermal diffusivity but they have certain disadvantages. In the pulse method, the entire knowledge of the thermal diffusivity is contained in the shape of the temperature response curve. Consequently, the recording facilities have to reproduce this shape as accurately as possible. The disadvantage of the temperature wave technique is that the phase shift or the amplitude ratio of temperature oscillations at

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two different points of a specimen has to be measured. In both cases, rather complicated instruments to achieve acceptable accuracy [3] have to be used. However, there is one more thermal diffusivity-dependent parameter—frequency, the measurement of which is easier and more accurate.

In the method proposed in this paper, the signal phase shift in the test specimen is invariable and equal to π, while the frequency of temperature oscillations depends on the thermal diffusivity of the specimen and can be measured with a high precision. The test specimen is an element of the thermal feedback in a control system, CS, where the phase delay takes place only in the specimen, and so the oscillations are excited at the frequency at which the phase shift in the specimen is π. The shift in an entire feedback loop is 2π because of the signal inversion in the controller.

The CS may lose stability if any one of its parameters changes in a particular way [4]. Such a parameter is referred to as bifurcational and its value at which stability is lost is referred to as critical. For certain kinds of system nonlinearity, the loss of stability may result in auto-oscillations with an amplitude $\xi(\varepsilon)$ at frequency $\omega(\varepsilon)$, where $\varepsilon = (A - A_c)/A_c$ is the relative departure of the generalized gain factor A, which depends on the bifurcational parameter, from its critical value A_c. This excitation mode is referred to as soft; $\xi(\varepsilon)$ tends to zero and $\omega(\varepsilon)$ to its critical value ω_c as $\varepsilon \to 0$.

If in a CS, whose feedback loop includes the test speciman, the soft mode is possible, auto-oscillations can be excited by varying A from zero to its critical value A_c. By measuring the auto-oscillation frequency, the thermal diffusivity of the test specimen can be obtained. The formula for calculation of the thermal diffusivity can be derived through analyzing the CS equation by asymptotic methods, which also yield the range $0 < \varepsilon < \varepsilon_0$ where the system may be regarded as linear, within the desired accuracy, and the relationship among the thermal diffusivity a, the frequency of the linear system ω_c, and the geometrical parameter C is the most simple:

$$a = C\omega_c \tag{1}$$

The method and its applications have been described previously [5], where the equation of the thermophysical CS was analyzed and Eq. (1) was obtained. The auto-oscillation excitation mode was not discussed. It was reported that a gradual reduction of the auto-oscillation amplitude to zero as $\varepsilon \to 0$, which is characteristic of a soft mode, and a weak dependence $\omega = \omega(\varepsilon)$ were observed. Consequently, in the zeroth approximation, the experimental auto-oscillation frequency $\omega(\varepsilon)$ was assumed to be equal to ω_c and so Eq. (1) became applicable.

The present paper is devoted to further analysis of the problem formulated earlier [5] preceded by a more thorough statement of the problem and linear analysis. In the section dealing with nonlinear analysis, the algorithm of the method is described in detail and the auto-oscillation excitation mode is shown to be really soft, while the frequency is weakly dependent on ε.

2. STATEMENT OF THE PROBLEM

A thermophysical CS can be obtained in numerous combinations of the test specimen shape, controller type, and boundary conditions, but auto-oscillations cannot be excited in all systems. Nevertheless, the algorithm of analysis of these different systems is the same and will be demonstrated for the most simple problem: CS with a flat specimen and a proportional controller. The system, shown in Fig. 1, consists of test specimen 1, thermostat 2, differential thermocouple 3, variable reference voltage source 4, amplifier 5, and heater 6. The test specimen is in thermal contact with the thermostat and heater, which generates a one-dimensional heat flux. One of the thermocouple's junctions is inside the specimen at point x_0 and the other is on the surface being thermostated, $x = 0$. The thermocouple signal is fed to the inverting input of the amplifier, whose other input is connected to the reference voltage source, and the output, to the heater. The test specimen, thermocouple, reference voltage source, amplifier, and heater make a closed-loop CS. The equation of this system

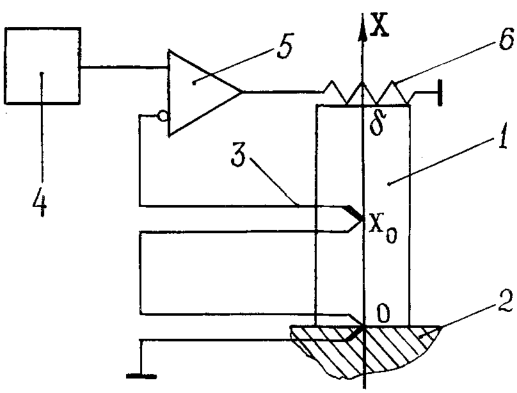

Fig. 1. A control system with a proportional controller and a flat specimen: 1, test specimen: 2, thermostat; 3, differential thermocouple; 4, reference voltage source; 5, controller; 6, heater.

is derived by eliminating intermediate variables u_1, u_2, and q from equations of CS elements

$$\left.\begin{array}{ll} u_1 = \beta T(x_0, t), & \text{thermocouple,} \\ u_2 = K(u_0 - u_1), & \text{controller,} \\ q = u_2^2 \sigma(u_2)/SR, & \text{heater,} \\ \dot{T}(x, t) = aT''(x, t), & \text{specimen} \\ T(x, t)|_{x=0} = 0; & \lambda T'(x, t)|_{x=\delta} = q \end{array}\right\} \quad (2)$$

The symbols are defined under Nomenclature, at the end of the paper. The system acquires the form of a nonlinear boundary value problem,

$$\left.\begin{array}{l} \dot{T}(x, t) = aT''(x, t) \\ T(x, t)|_{x=0} = 0 \\ \lambda T'(x, t)|_{x=\delta} = (K^2/SR)[u_0 - \beta T(x_0, t)]^2 \, \sigma[u_0 - \beta T(x_0, t)] \end{array}\right\} \quad (3)$$

As a rule, $T(x, t)$ is within 1 or 2 K, so the linear approximation for the thermocouple relation in Eq. (2) is sufficient.

In fact, analysis of the CS equation amounts to solving a nonlinear problem of nonstationary heat conduction, which is done in two stages. First, solution of the linearized problem expressed by Eq. (3) is examined, then the problem is subjected to nonlinear analysis.

3. LINEAR ANALYSIS

Let us linearize the problem expressed by Eq. (3) in the vicinity of its stationary solution. We shall concentrate on periodic solutions of the linearized problem, because a can be determined from the parameters of stable auto-oscillations only. These solutions are obtained by the Fourier method; their spatial parts will be the eigenfunctions of some operator. The eigenvalues of this operator will be the squared wavenumbers of the temperature waves excited in the specimen. Solving the problem for the eigenvalues, we arrive at Eq. (1), which relates the auto-oscillation frequency to the thermal diffusivity of the test specimen. We solve the problem expressed by Eq. (3) as a sum of steady-state $\bar{T}(x)$ and periodic $\tilde{T}(x, t)$ solutions $T(x, t) = \bar{T}(x) + \tilde{T}(x, t)$. The steady-state solution is

$$\left.\begin{array}{l} \bar{T}(x) = (u_0/\beta x_0)[D \pm (D^2 - 1)^{1/2}]\, x \\ \text{where} \quad D = 1 + \lambda SR/2\beta u_0 x_0 K^2 \end{array}\right\} \quad (4)$$

Of the two solutions of Eq. (4), we choose the one for which the Heaviside function's argument is positive,

$$\sigma[u_0 - \beta \bar{T}(x_0)] = 1, \qquad u_0 > \beta \bar{T}(x_0) \tag{5}$$

i.e.,

$$\bar{T} = (u_0/\beta x_0)[D - (D^2 - 1)^{1/2}]\,x \tag{6}$$

Linearizing the problem expressed by Eq. (2) in the vicinity of the stationary solution given by Eq. (6), i.e., neglecting terms in $\tilde{T}^2$, we have

$$\left.\begin{aligned} &\dot{\tilde{T}}(x, t) = a\tilde{T}''(x, t) \\ &\tilde{T}(x, t)|_{x=0} = 0, \qquad \tilde{T}'(x, t)|_{x=\delta} = -\frac{2\beta K^2}{\lambda SR}[u_0 - \beta \bar{T}(x_0)]\,\tilde{T}(x_0, t) \end{aligned}\right\} \tag{7}$$

After substitution of Eq. (6) to the boundary condition of Eq. (7), the latter acquires the form

$$\tilde{T}'(x, t)|_{x=\delta} = [1 - \sqrt{(D+1)/(D-1)}]\,\frac{\tilde{T}(x_0, t)}{x_0} = \frac{A}{x_0}\,\tilde{T}(x_0, t) \tag{8}$$

Parameter $A = 1 - [(D+1)/(D-1)]^{1/2}$ depends on the system parameters determining the gain of the entire feedback loop. Two of the parameters buried in A, the reference voltage u_0 and the amplifier gain K, are variables. An auto-oscillation can be excited by varying either of them from zero to their critical value. According to the above-mentioned terminology, A is a generalized gain factor, while u_0 and K are bifurcational parameters. Expressing D from $A = 1 - [(D+1)/(D-1)]^{1/2}$ and substituting to Eq. (6) we obtain

$$\bar{T}(x) = \frac{u_0}{\beta}\left[\frac{A}{A-2}\right]\frac{x}{x_0} \tag{9}$$

where $A = 1 - [1 + 4\beta x_0(u_0 K^2)/\lambda SR]^{1/2}$ depends on both bifurcational parameters. It is easy to see that the excitation of auto-oscillation by u_0 adjustment is hardly possible, as it would result in $\bar{T}(x)$ increase to unpredictable limit. On the contrary, K is a very convenient parameter for auto-oscillation excitation and control of their amplitude. Though $\bar{T}(x)$ increases together with K, it never exceeds a certain limit $\bar{T}(x) = u_0 x/\beta x_0$.

The nonstationary solution of the problem (7) will be tried as $\tilde{T}(x, t) = V(x)\exp(ak^2 t)$. Its spatial part, $V(x)$, is the eigenfunction of the operator

$$\left.\begin{aligned} &V''(x) = k^2 V(x) \\ &V(x)|_{x=0} = 0, \qquad V'(x)|_{x=\delta} = \frac{A}{x_0}\,V(x_0) \end{aligned}\right\} \tag{10}$$

It is obvious that the operator in Eq. (10) has a digital spectrum of eigenfunctions,

$$V_j(x) = C_j \operatorname{sh}(k_j x) \tag{11}$$

and eigenvalues k_j satisfying the latter boundary condition of the operator in Eq. (10),

$$k x_0 \operatorname{ch}(k\delta) = \frac{A}{x_0} \operatorname{sh}(k x_0) \tag{12}$$

The roots of Eq. (12) depend on A, which varies from 0 to $-\infty$ with K being varied from 0 to ∞. The investigation of the roots of Eq. (12) in a complex plane reveals that with $A = 0$ all roots are imaginary, i.e., k_j^2 are real and negative. The corresponding nonstationary solution describes temperature relaxation from its initial distribution to steady-state $\bar{T}(x) = 0$. With K increase (A decrease), the pairs of roots start moving toward each other along an imaginary axis. With a certain $A(K)$ value the first pair (we enumerate them in the increasing order $k_j < k_{j+1}$) collapses and goes on moving in a complex plane as a single complex root $k_1' + ik_1''$. As soon as it leaves the imaginary axis, the convergent periodic solution

$$\tilde{T}_1(x, t) = C_1 \operatorname{sh}(k_1' + ik_1'')x \exp[a(k_1'^2 - k_1''^2)t] \exp(i2k_1'k_1''at) \tag{13}$$

appears. Further K increase results in a dominant k_1' over k_1'' increase and a respective $k_1'^2 - k_1''^2$ reduction, until $k_1' = k_1'' = k_1$ and a stable periodic solution;

$$\tilde{T}(x, t) = C \operatorname{sh}(1+i)k_1 x \exp(i2ak_1^2 t) \tag{14}$$

is established. The values of $K = K_c$ and $A = A_c$ that correspond to the condition $k_1' = k_1'' = k_1$ are defined as "critical." In the supercritical domain when $A < A_c$ the diverging solution appears due to $k_1' > k_1''$ and the linear analysis based on the assumption $\tilde{T}(x_0, t) \ll 2[u_0/\beta - \bar{T}(x_0)]$ is no longer valid.

The solution given by Eq. (14) implies that a is related to the oscillation frequency of the linear system ω_c as

$$a = \omega_c / 2k_1^2 \tag{15}$$

To obtain the equation for k_1 calculation, let us substitute $(1+i)k$ in Eq. (12). Introducing notations $v = k\delta$, $\eta = x_0/\delta$, and separating the real and imaginary parts of Eq. (12), we have

$$\begin{aligned} \eta v(\operatorname{ch} v \cos v - \operatorname{sh} v \sin v) &= A \operatorname{sh} \eta v \cos \eta v \\ \eta v(\operatorname{ch} v \cos v + \operatorname{sh} v \sin v) &= A \operatorname{ch} \eta v \sin \eta v \end{aligned} \tag{16}$$

Eliminating the parameter A in Eq. (16), we obtain an equation for eigenvalues of operator (10),

$$\frac{\operatorname{ch} v \cos v + \operatorname{sh} v \sin v}{\operatorname{ch} \eta v \sin \eta v} = \frac{\operatorname{ch} v \cos v - \operatorname{sh} v \sin v}{\operatorname{sh} \eta v \cos \eta v} \tag{17}$$

It is easy to see that k is a wavenumber of temperature wave, while $k_1(\delta - x_0) = v_1(1-\eta)$ is a phase delay of harmonic feedback signal in a specimen. The complicated form of the phase condition expressed by Eq. (17) is the result of superposition of two temperature waves running to and reflected from the thermostat. If the reflected wave buried in Eq. (14) is neglected, then $\tilde{T}(x, t) = (C/2)\exp(k_1 x)\exp i(\omega t + k_1 x)$ and Eq. (17) reduces to

$$tg(1-\eta)v = -1 \tag{18}$$

which implies that phase delay in a specimen $(1-\eta)v_1 = 3\pi/4$. The residual of phase delay $2\pi - (1-\eta)v_1$ is distributed between controller π, as it inverts the feedback signal, and heater $\pi/4$, whose temperature oscillation $\tilde{T}(\delta, t)$ delays from that of power $q(t)$.

The thermal diffusivity can be obtained from Eq. (15) by measuring the auto-oscillation frequency $\omega(\varepsilon)$ of the CS within $0 < \varepsilon \ll 1$, i.e., when the system nonlinearity is a priori small and $\omega(\varepsilon) = \omega_c$. The wavenumber k_1 for a specified η is found from numerical solution of Eq. (17), while the corresponding critical value A_c is obtained by k_1 substitution to any of Eqs. (16). Figure 2 displays the dependences $v_1(\eta)$ and A_c (η) in the span $0.1 \leqslant \eta \leqslant 0.9$. With $\eta \to 0$, Eq. (17) tends to the form $\operatorname{sh}(v)\sin(v) = 0$, hence $v_1 \to \pi$ and $A_c \to -11.5920$. As with the thermocouple being approached to the

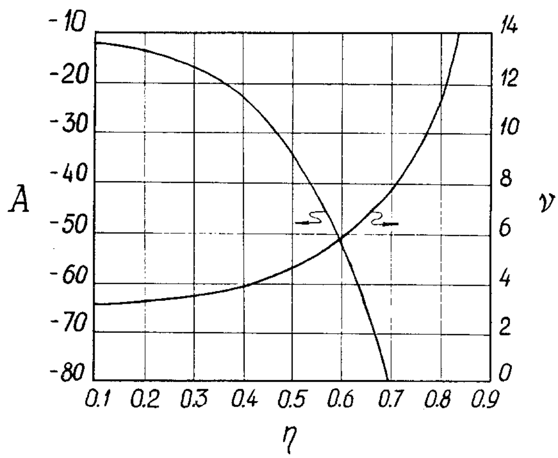

Fig. 2. Dependence of generalized gain A_c and phase delay v_1 on thermocouple position η.

heater ($\eta \to 1$), the root ν_1 tends to infinity, to satisfy the phase condition $(\eta - 1)\nu_1 = 3\pi/4$, so does the autooscillation frequency. With respect to short temperature wave intensive decay, the generalized gain A_c should decrease with $\eta \to 1$ as shown in Fig. 2. At $\eta = 1$ the auto-oscillations cannot be excited by any means; thus the utilization for this purpose of a thermal probe (heater and thermocouple joined together) in combination with a proportional controller is impossible. In our experiments the thermocouple was fixed in the middle of the specimen: $\eta = 0.5$; $\nu_1 = 4.6941$; and $A_c = -34.6915$.

The thermal diffusivity measuring procedure consists of the following stages. First, auto-oscillations are excited by gradualy increasing K, while u_0 is fixed. Then, with adjustment of u_0, the desired temperature gradient [see Eq. (9)] is obtained. Finally, the temperature oscillation amplitude is established by fine regulation of K. If the temperature dependence of the thermal diffusivity $a(T)$ has to be measured, the most convenient way is to raise the temperature of the thermostat monotonically, with simultaneous recording of the temperature oscillations. The only experimental parameter, ω_c (or oscillation period), required for the a calculation is obtained as the ratio of the time span containing an integer number of oscillations: the number of oscillations.

4. NONLINEAR ANALYSIS

As noted above with $A < A_c$ (A always < 0) the exponent index $k_1'^2 - k_1''^2$ in Eq. (13) changes sign and an exponentially increasing solution appears. This would result in an infinite increase in the oscillation amplitude, but this does not occur because of the constraining effect of the real system nonlinearity. As the root of Eq. (12) moves in a complex plane, its imaginary, as well as its real part, varies so that $\omega \neq \omega_c$. If the autooscillation excitation mode is soft, then, as ε increases, so does the difference between the experimental frequency $\omega(\varepsilon)$ and ω_c, and the applicability range of Eq. (15) has to be questioned. We stress that this question is valid only in the case of a soft oscillation excitation mode. Let us show that the mode is soft indeed and derive the relationship between a and $\omega(\varepsilon)$.

For this purpose the nonstationary part of the problem expressed by Eq. (3) is represented in the form

$$\left.\begin{aligned} \dot{W}(x, t) &= aW''(x, t) \\ W(x, t)|_{x=0} &= 0 \\ W'(x, t)|_{x=\delta} &= \frac{A}{x_0}(1+\varepsilon)\, W(x_0, t) + W^2(x_0, t) \end{aligned}\right\} \tag{19}$$

where $\varepsilon=(A-A_c)/A_c$ is the relative departure of A from A_c; $W(x,t)=(K^2\beta^2/\lambda SR)\,\tilde{T}(x,t)$. Assume also that

$$\begin{aligned} &t=(1+c)\tau; \qquad c=c_2\xi^2+c_4\xi^4+\cdots; \qquad \varepsilon=b_2\xi^2+b_4\xi^4+\cdots \\ &W(x,\tau)=\xi W_1(x,\tau)+\xi^2 W_2(x,\tau)+\cdots \end{aligned} \tag{20}$$

where ξ is the amplitude of the linear term and $W_2, W_3,\ldots$ are trigonometric polynomials with x-dependent coefficients. The function W_{2n} contains only even harmonics whose ordinal number does not exceed $2n$, while W_{2n+1} contains only odd harmonics.

Substituting Eq. (20) into Eq. (19) and equating the expressions with identical powers of ξ, we have a sequence of linear inhomogeneous boundary value problems for determining the unknown functions $W_2, W_3,\ldots$ and constants $c_2, c_4,\ldots$ and $b_2, b_4,\ldots$ The unknown functions W_n are determined recurrently by solving these boundary value problems, while the constants c_{2n} and b_{2n} are determined at the $2n+1$st step of the algorithm from conditions of resolvability in the class of trigonometric polynomials of the corresponding boundary value problems. If it is found that $b_2>0$, then with $A<A_c$ (or with $\varepsilon>0$), the stable periodic solution bifurcates from the equilibrium state, which, over any time span of order ε^{-1}, behaves asymptotically as

$$\begin{aligned} W(x,t)=(\varepsilon/b_2)^{1/2}[&\mathrm{sh}(1+i)kx\exp(i\omega\tau)+\mathrm{sh}(1-i)kx\exp(-i\omega\tau)] \\ &+(\varepsilon/b_2)\,W_2(x,\tau)+o(\varepsilon^{3/2}) \end{aligned} \tag{21}$$

This is the soft auto-oscillation excitation mode: the equilibrium state loses stability but a stable periodic solution is obtained in its vicinity. If, however, $b_2<0$, then a periodic solution with Eq. (21) asymptotic behavior exists with $\varepsilon>0$, in the precritical domain. In this case, the auto-oscillations acquire from the start an amplitude with which the problem is essentially nonlinear, and none of the asymptotic methods is applicable. If, however, $b_2=0$, the entire reasoning is reiterated with b_4, etc.

Because it turned out that $b_2>0$, let us limit the recurrent sequence of inhomogeneous boundary value problems to the first three,

$$\left.\begin{aligned} &\dot{W}_1(x,\tau)=aW_1''(x,\tau) \\ &W_1(x,\tau)|_{x=0}=0 \\ &W_1'(x,\tau)|_{x=\delta}=(A_c/x_0)\,W_1(x_0,\tau) \end{aligned}\right\} \tag{22}$$

$$\left.\begin{aligned} &\dot{W}_2(x,\tau)=aW_2''(x,\tau) \\ &W_2(x,\tau)|_{x=0}=0 \\ &W_2'(x,\tau)|_{x=\delta}=(A_c/x_0)\,W_1(x_0,\tau)+W_1^2(x_0,\tau) \end{aligned}\right\} \tag{23}$$

$$\left.\begin{aligned}&\dot{W}_3(x,\tau)=aW_3''(x,\tau)+ac_2W_1''(x,\tau)\\&W_3(x,\tau)|_{x=0}=0\\&W_3'(x,\tau)|_{x=\delta}=(A_c/x_0)\,[W_3(x_0,\tau)+b_2W_1(x_0,\tau)]\\&\qquad+2W_1(x_0,\tau)\,W_2(x_0,\tau)\end{aligned}\right\}\tag{24}$$

The problem expressed by Eq. (22) is easily seen to coincide with Eq. (7), whose periodic solution has been discussed above. To make the parameters of the expansion given by Eq. (20) dimensionless, the solution of Eq. (22) is represented in the form

$$W_1(x,\tau)=W_0[\mathrm{sh}(1+i)kx\exp(i\omega\tau)+\mathrm{sh}(1-i)kx\exp(-i\omega\tau)]\tag{25}$$

where $W_0=1\ \mathrm{m}^{-1}$. Substituting (25) in the last boundary condition of the problem expressed by Eq. (23), we find that the inhomogeneity contains the zero and second harmonics. The solution is tried in the same form as the inhomogeneity, $W_2(x,\tau)=P(x)\exp(i2\omega\tau)+\bar{P}(x)\exp(-i2\omega\tau)+Q(x)$. As a result, we have

$$\left.\begin{aligned}&P(x)=\frac{W_0^2\,\mathrm{sh}^2(1+i)kx_0\,\mathrm{sh}\sqrt{2}(1+i)kx_0}{\sqrt{2}(1+i)k\,\mathrm{ch}\sqrt{2}(1+i)\,k\delta-(A_c/x_0)\,\mathrm{sh}\sqrt{2}(1+i)\,kx_0}\\&Q(x)=\frac{2W_0^2}{1-A_c}\,[\mathrm{sh}(1+i)\,kx_0\,\mathrm{sh}(1-i)\,kx_0]\,x\end{aligned}\right\}\tag{26}$$

Substitution of Eqs. (25) and (26) in Eq. (24) produces the first and the third harmonics in the inhomogeneity. Assuming that $W_3(x,\tau)=L(x)\exp(i\omega\tau)+\bar{L}(x)\exp(-i\omega\tau)+M(x)\exp(i3\omega\tau)+\bar{M}(x)\exp(-i3\omega\tau)$ and equating the terms of identical harmonics, we obtain two problems,

$$\left.\begin{aligned}&L(x)=L''(x)/2ik^2+c_2W_0\,\mathrm{sh}(1+i)\,kx\\&L(x)|_{x=0}=0\\&L'(x)|_{x=\delta}=(A_c/x_0)[L(x_0)+b_2W_0\,\mathrm{sh}(1+i)\,kx_0]\\&\qquad+2P(x_0)\,W_0\,\mathrm{sh}(1-i)\,kx_0+2Q(x_0)\,W_0\,\mathrm{sh}(1+i)\,kx_0\end{aligned}\right\}\tag{27}$$

$$\left.\begin{aligned}&M(x)=i6k^2M''(x)\\&M(x)|_{x=0}=0\\&M'(x)|_{x=\delta}=(A_c/x_0)\,M(x_0)+2P(x_0)\,W_0\,\mathrm{sh}(1+i)\,kx\end{aligned}\right\}\tag{28}$$

The problem expressed by Eq. (28) is always resolvable, while the condition of resolvability of Eq. (27) is

$$(m_1+im_2)c_2+(n_1+in_2)b_2+[2/(1-A_c) - W_0^2(l_1+il_2)/(z_1^2+z_2^2)](P_1+iP_2)=0 \tag{29}$$

where

$$m_1 = -(v_c^2/\delta)\operatorname{ch} v_c \sin v_c + (A_c v_c/2\delta)(\operatorname{sh}\eta v_c \sin\eta v_c - \operatorname{ch}\eta v_c \cos\eta v_c) + \frac{n_1}{2}$$

$$m_2 = (v_c^2/\delta)\operatorname{sh} v_c \cos v_c - (A_c v_c/2\delta)(\operatorname{sh}\eta v_c \sin\eta v_c - \operatorname{ch}\eta v_c \cos\eta v_c) + \frac{n_2}{2}$$

$$\begin{aligned}
&n_1=(A_c/x_0)\operatorname{sh}\eta v_c \cos\eta v_c, \qquad n_2=(A_c/x_0)\operatorname{ch}\eta v_c \sin\eta v_c\\
&z_1=\sqrt{2}\,\eta v_c(\operatorname{ch}\sqrt{2}\,\eta v_c \cos\sqrt{2}\,\eta v_c - \operatorname{sh}\sqrt{2}\,\eta v_c \sin\sqrt{2}\,\eta v_c)\\
&\qquad - A_c \operatorname{sh}\sqrt{2}\,\eta v_c \cos\sqrt{2}\,\eta v_c\\
&z_2=\sqrt{2}\,\eta v_c(\operatorname{ch}\sqrt{2}\,\eta v_c \cos\sqrt{2}\,\eta v_c + \operatorname{sh}\sqrt{2}\,\eta v_c \sin\sqrt{2}\,\eta v_c)\\
&\qquad - A_c \operatorname{ch}\sqrt{2}\,\eta v_c \sin\sqrt{2}\,\eta v_c\\
&P_1=2x_0 W_0^2(\operatorname{sh}^2\eta v_c+\sin^2\eta v_c)\operatorname{sh}\eta v_c \cos\eta v_c\\
&P_2=2x_0 W_0^2(\operatorname{sh}^2\eta v_c+\sin^2\eta v_c)\operatorname{ch}\eta v_c \sin\eta v_c\\
&l_1=z_1 \operatorname{sh}\sqrt{2}\,\eta v_c \cos\sqrt{2}\,\eta v_c + z_2 \operatorname{ch}\sqrt{2}\,\eta v_c \sin\sqrt{2}\,\eta v_c\\
&l_2=z_1 \operatorname{ch}\sqrt{2}\,\eta v_c \sin\sqrt{2}\,\eta v_c - z_2 \operatorname{sh}\sqrt{2}\,\eta v_c \cos\sqrt{2}\,\eta v_c
\end{aligned} \tag{30}$$

Separation of the real and imaginary parts of Eq. (29) gives c_2 and b_2. Thus for $\eta=0.5$ the Lyapunov coefficients c_2 and b_2 take on the values

$$c_2=0.010569x_0 W_0^2, \qquad b_2=0.066410x_0 W_0^2, \qquad c_2/b_2=0.15915 \tag{31}$$

Hence $b_2>0$, the auto-oscillation excitation mode is soft, and as $\varepsilon \to 0$, an emplitude of periodic solution,

$$\begin{aligned}
\tilde{T}(x,t)=\frac{2u_0x_0}{\beta}[D(\varepsilon)-1]\Big\{&\sqrt{\varepsilon/b_2}\,2W_0(\operatorname{sh}^2 kx+\sin^2 kx)^{1/2}\\
&\times\cos[\omega(\varepsilon)t+\phi(x)]+\operatorname{Re}(\varepsilon/b_2)\\
&\times[P(x)\exp(i2\omega(\varepsilon)t)+\bar{P}(x)\exp(-i2\omega(\varepsilon)t)]\Big\}\\
&+o(\varepsilon^{3/2}), \qquad \phi(x)\operatorname{arctg}(\operatorname{cth} kx \operatorname{tg} kx)
\end{aligned} \tag{32}$$

tends to zero.

In a real oscillating system $\varepsilon \neq 0$ and the relation between ω_c and $\omega(\varepsilon)$, which is analogous to $t = (1 + c)\tau$, in the first approximation is

$$\left.\begin{aligned} &\omega(\varepsilon) = [1 + (c_2/b_2)\varepsilon]\,\omega_c \\ \text{and for } \eta = 0.5,\quad &\\ &\omega(\varepsilon) = (1 + 0.15915\varepsilon)\,\omega_c \end{aligned}\right\} \tag{33}$$

Substituting for ω_c in Eq. (15), we obtain the improved formula for the thermal diffusivity calculation:

$$\left.\begin{aligned} &a = \omega(\varepsilon)/2k_1^2[1 + (c_2/b_2)\varepsilon] \\ \text{and for } \eta = 0.5,\quad &\\ &a = 2.2692\ 10^{-2}\omega(\varepsilon)\delta^2/(1 + 0.15915\varepsilon) \end{aligned}\right\} \tag{34}$$

5. ERRORS OF THE METHOD

Though relations given by Eq. (34) approximate the relation between a and $\omega(\varepsilon)$ better than Eq. (15), their practical application, requiring the A experimental determination, is rather useless, as the main advantages of the method proposed are lost. Nevertheless, these expressions permit an estimation of the error that is made if the experimental frequency $\omega(\varepsilon)$ is substituted into Eq. (15) instead of Eq. (34). Let us denote the respective values of thermal diffusivity a_c and a_ε. Then the error due to CS nonlinearity is

$$\Delta a/a_\varepsilon = (a_\varepsilon - a_c)/a_\varepsilon = (c_2/b_2)\varepsilon \tag{35}$$

The upper limit of the error can be estimated from the condition $\sigma(u_2) = 1$, which implies that

$$\left.\begin{aligned} &u_2 = K[u_0 - \beta\bar{T}(x_0, \varepsilon) - \beta\tilde{T}(x_0, t, \varepsilon)] > 0 \\ \text{and for } \eta = 0.5\quad &\\ &0.05773 - 0.01926\varepsilon - \sqrt{\varepsilon}\ 0.13525 \cos\omega(\varepsilon)t > 0 \end{aligned}\right\} \tag{36}$$

Consequently ε cannot exceed a certain value ε_0 ($\varepsilon_0 = 0.16271$ for $\eta = 0.5$) with which an amplitude of harmonic component $\beta\tilde{T}(x_0, t, \varepsilon)$ is equal to $u_0 - \beta\bar{T}(x_0, \varepsilon)$. The voltage fed to recording facilities is proportional to the so-called error signal u_2, consisting of constant and alternating components. As both of them depend on ε, the latter determines the depth of recording voltage modulation. We usually use 50% modulation to guarantee stability of auto-oscillations and signal transition linearity even in the case of a sudden increase in its amplitude (at specimen phase transition, for example). The respective value of ε is referred to as optimal and can be evaluated from the condition of steady component $u_0 - \beta\bar{T}(x_0, \varepsilon)$ equality to the double amplitude of alternating component $\beta T_0(x_0, t, \varepsilon)$. Substituting Eq. (9) and an amplitude of the first harmonic of temperature oscillation $T_0(x_0, t, \varepsilon)$, expressed from Eq. (32), to $u_0 - \beta\bar{T}(x_0, \varepsilon) = 2\beta\tilde{T}(x_0, t, \varepsilon)$, we obtain

$$-A_c(1+\varepsilon) = 8x_0 W_0 \sqrt{(\varepsilon/b_2)(\mathrm{sh}^2\,\eta v_1 + \sin^2 \eta v_1)} \tag{37}$$

The analogous condition for $\eta = 0.5$ can be obtained from the second of Eqs. (36), relating the steady component to ε more accurately than Eq. (37),

$$0.0577 - 0.0192\varepsilon = 2\sqrt{\varepsilon}\, 0.1353 \tag{38}$$

Calculating $\varepsilon_{opt} = 0.0470$ and substituting this value into the formula for error due to nonlinearity expressed by Eq. (35), we find that $(\Delta a_\varepsilon/a)100\% = 0.75\%$. The lower limit of ε depends on oscillation stability and sensitivity of recording instruments. Actually the modulation depth can be reduced to at least 10% without loss of stability and $\omega(\varepsilon)$ measurement accuracy. The analysis of other systematic errors has been given in the literature [5]. The main contributing error $\Delta a_v/a = 2.07\Delta x_0/\delta$, resulting from thermocouple coordinate uncertainty Δx_0, is essential to all contact temperature measurement. The total value of all other contributing errors of the method is negligible.

NOMENCLATURE

A	Generalized gain factor
A_c	Critical value of A
a	Thermal diffusivity
b_2, c_2	Lyapunov coefficients
K	Controller gain
k	Wavenumber
q	Heat flux
R	Heater resistance
S	Heater surface area
$T(x, t)$	Difference between specimen and thermostat temperatures
t	Real time
u_0	Reference voltage
u_1, u_2	Voltage
x	Coordinate

x_0	Thermocouple coordinate
β	Thermo emf factor
δ	Specimen thickness
ε	Relative deflection of A from A_c
η	Thermocouple normalized coordinate
λ	Thermal conductivity
ν	Temperature wave phase delay through the whole specimen
ξ	Auto-oscillation amplitude
σ	Heaviside step function
τ	Normalized time
ϕ	Spatial part of phase
ω	Frequency

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